A GENERATING FUNCTION WITH MAXIMUM COMBINATION OF ROOK AND BISHOP MOVEMENT ON A TWO-DIMENSIONAL CHESS BOARD IS ALSO A MAXIMUM QUEEN MOVEMENT

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Abstract
The techniques used by chess players to checkmate vary greatly, most often, players have different combinations of the following; Queen, Knights, Bishops, Rooks and Pawns to protect the King from any attack by the opponent player on an 8×8 board that is played by two players. However, focusing on a combination of rook and bishop moves which is like the movement of the Queen on a chess board to study combined movement within the forbidden space. All the same, we constructed the rook and bishop movement generating function with a better computational result to that of the queen generating function on a chessboard with forbidden space. Furthermore, we applied these techniques used in the construction of rook and bishop generating functions to solve problems for different cases on disjoined sub-boards.

Keywords: Chess movements; Permutation; r-arrangement; combinatorial structures; Disjoined sub boards; generating function; n-queen puzzle.

1.0 Introduction

Many chess players apply different techniques to checkmate their opponent. Most often, player have different combinations of the following; Queen, knights, bishops, rooks and pawns to protect the King from any attack by the opponent player on an 8×8 board that is played by two players. The Queen is a great piece with a crown on its head and is placed beside the king next to the bishop on the central square that matches the piece's colour. In addition, the queen has a very high value that is worth nine pawns for any good chess player to accept it to be exchanged for any other piece. Each player starts with one queen piece. Although, a pawn can be conditionally transformed into a queen. However, a combination of rook and bishop moves is like the movement of the Queen on a chess board where, she moves forward or backward or left or right or diagonally along its path if there is no obstruction by another piece. The queen can be used to capture any of her opponent's pieces on the board.

In addition, the eight queens puzzle is a chess board problem that is generated by placing a maximum number of non-attacking queens on an 8×8 chess board (Rok Sosic and Jun Gu, 1990). However, in solving the problem, one need not place any two queens such that there can share the same diagonal, row or column, that is called non-attacking queens board. Thus, the chess board puzzle for eight queens gives a more general problem for n-queens placement on an $n \times n$ board for n non-attacking queens for all $n > 3$ (Hoffman et al., 1969; *Bo*

Bernhardsson 1991; Barr and Rao, 2006). In 1848 Max Bezzel, the chess composer published the puzzle with eight queens while Franz Nauck, in 1850 published the first solution (Rouse Ball, 1960). In addition, Nauck in his paper extended the eight queens puzzle to the n-queens problem on an $n \times n$ chess board. Subsequently, many authors including Carl Friedrich Gauss, have made positive contributions on eight queens and n-queens puzzle. S. Gunther in 1874, constructed the technique of applying the determinants to find the solution of the given puzzle. While Gunther's approach was further refined by J, W. L. Glaisher (Rouse Ball, 1960).

Likewise, in 1972, Edsger Dijkstra applied the eight queens and n-queens puzzle to show the power of structured programming. In addition, he published a very important and detailed explanations of a depth-first backtracking algorithm (Rouse Ball,1960)

However, in this problem we are looking for the maximum number of queens and their possible arrangements on an 8×8 board is given by $\binom{64}{8} = 4{,}426{,}165{,}368$ ways which is absolutely expensive when carrying out its' computation that give only ninety-two solutions. However, the possibility of using shortcut techniques that reduce the computational rules of thumb trying not to meet with the brute-force computational techniques. Thus, by applying a technique that is simple and constrain the maximum number of queens each to a single horizontal row (or vertical column), even though it is still brute force, we have possibilities to reduce the number of arrangements to $8^8 = 16{,}777{,}216$. Generating permutations can further reduce the arrangements to $8! = 40{,}320$ possibilities with a check for diagonal attacks.

Furthermore, with pieces different from queen on an 8×8 board such as knights can be placed in 32 places, or bishops in 14 places, kings in 16 places or eight rooks in 8places, so that no two pieces attack each other (Laisin, Okoli, & Okaa-onwuogu, 2019; Laisin & Uwandu, 2019; Laisin, 2018; LAISIN, 2018). In a game of chess, pieces can also been substituted for a queen once it gets to the last row i.e. row 8 for white pieces or row 1 for black pieces. In the case of knights, it is easy to place each one on an opposite colour square, since the movement of attack denoted as $\Gamma \, or \, \mathrm{T}$ i.e. moving and attacking on opposite color. The rooks and king's placement is very easy for placing eight rooks along a diagonal (i.e. one among thousands of other solutions), and a total of 16 kings can be placed on the chess board by sub dividing the board into 2×2 sub boards and placing the kings at equal distances on each 2×2 sub board.

The Bishop is a tall slender piece with pointed tip that has a strange cut made into it and it sits next to the knight piece. It has a value that is less than that of a rook. The bishop movement can attack pieces as many unoccupied squares as possible diagonally as far as there is no piece hindering its path movement. Bishops capture opposing pieces by landing on the square occupied by an opponent piece. A bishop potential is maximized by placing it on an open, long diagonal such that it will not be obstructed by friendly pawn or an opponent's piece. A quick development of the bishop can be achieved by a special move called fianchetto. How a bishop gets along with pawns determines if it is a good or bad bishop. If your bishop and most of your pawn are on the same color squares then it is a bad bishop because it has fewer squares available to it. Each player starts out with two bishop pieces, each one residing on its own color of square. In addition, a bishop moves diagonally and captures a piece if that piece rests on a square in the same diagonal (LAISIN, 2018; Laisin, and Uwandu, 2019). However, the polynomial for nonattacking rook or bishop has a very good part to play in the theory of permutations with forbidden positions (Laisin, Okoli, & Okaa-onwuogu, 2019; Laisin & Uwandu, 2019; Laisin, 2018; LAISIN, 2018; Skoch, 2015; Jay & Haglund, 2000; Herckman, 2006; Chung, & Graham,1995) have shown that polynomial of either the bishop or rook on a given board can be generated recursively by applying cell decomposition techniques of Riordan (Abigail, 2004; Riordan, 1980; Riordan, 1958). Though, Laisin, Okoli, & Okaa-onwuogu, 2019; Laisin & Uwandu, 2019; LAISIN, 2018; Laisin & Ndubuisi, 2017; Jay Goldman, and James Haglund, 2000 studied, examined and investigated movement of bishop or rook on a chess-board with forbidden area to develop independent techniques for polynomials by applying generating functions. However, the generating functions for three-dimensional cases has been investigated for movement of rook on a chess-board with forbidden area (Laisin, Okeke, Chukwuma, 2020; Laisin, Chukwuma, Okeke, 2020; Zindle, 2007).

Furthermore, for a combination of rook and bishop moves in solving the maximum problem, one need not place either rook or bishop such that they cannot share the same diagonal, row or column, on a board, this is called non-attacking rook and bishop board. Notwithstanding, the chessboard puzzle for a maximum combination of rook and bishop gives a more general problem for n-rooks and n-bishops placement on an $n \times n$ chessboard for n non-attacking rooks and bishops combinations for all $n > 3$.

Now, we shall be focusing on the two-dimensional board for non-attacking rook and bishop placement to generate a rook and bishop polynomial within the forbidden space. Furthermore, we shall apply the techniques constructed for the rook and bishop generating functions to solve problems different cases on disjoined sub-boards.

2.0 Basic definitions

2.1 Variations on an $n \times n$ board

chess variations with related problems such as shogi can come to play. For example, the $n + k$ dragon kings problem to determine k shogi pawns and $n + k$ mutually non-attacking dragon kings on a shogi board of $n \times n$ (*Chatham, Doug, 2018).*

2.2 n-Permutation matrix

Geometrically, a permutation matrix is a set of n points lying on an $n \times n$ board just like the squares on an $n \times n$ chessboard, with row or column giving the address of each one point. Thus, the order of the n-permutation matrix gives the solution to the n-rooks puzzle (Laisin, 2018)

2.3 Domination

The minimum number of queens is the domination number given on an $n \times n$ board, that can occupy by moving through all the forbidden space. For example, $n = 8$ gives the queen's domination number to be 5-queens. (Demirörs, Rafraf, and Tanik. 1992; Gent, Jefferson, Christopher; Nightingale, Peter, 2017)

2.4 Queens and other pieces

In placement, one can mix pieces and this mixing is called Variants e.g. the mixing of queens with other pieces; for example, placing *m* queens and *m* knights on an $n \times n$ board so that no piece attacks another (Gent, Ian P.; Jefferson, Christopher; Nightingale, Peter, 2017) or by the placement of queens and pawns such that, non-attacking queen attack pawn (Hoffman et al., 1969; Rouse Ball 1960; Bernhardsson 1991)

2.5 Latin squares

The placement of each digit x through n-ways and in n positions on an $n \times n$ matrix such that no two instances of the same digit are in the same row or column.

2.6 Rook:

A rook is a chess piece that moves horizontally or vertically and can take (or capture) a piece if that piece rests on a square in the same row or column as the rook [5, 7, 8].

a. Board: A board B is an $n \times m$ array of n rows and m columns. When a board has a darkened square, it is said to have a forbidden position.
b. Rook polynomial: A rook polynomial on a board B, with forbidden positions is denoted as $R(x, B)$, given by

$$R(x, B) = \sum_{i=1}^{k} r_i(B)x^i$$

where $R(x, B)$ has coefficients $r_i(B)$ representing the number of non-capturing rooks on board B. Clearly, we have just one way of not placing a rook. Thus $r_0(B) = 1$

c. A board B with forbidden positions, is said to be disjoint if the board can be decomposed into two sub-boards $B_i : i = 1 \ and\ 2$such that, neither B_1nor B_2share the same row or column. (Laisin, 2018)

2.7 Definition

Suppose that B be is an $m \times m$ board and its diagonal denoted by $\mathfrak{D}^\theta$ and let;

$$F(y_1, y_2, \dots, y_k) = \sum f(m_1, m_2, \dots, m_k) y_1^{m_1} y_2^{m_2} \dots y_k^{m_k} \in K[[(y_1, y_2, \dots, y_m)]]$$

Then, the $\mathfrak{D}^\theta$ is the power series in a single variable y defined by

$$\mathfrak{D}^\theta = \mathfrak{D}^\theta(y) = \sum_m f(m, m, \dots, m) y^m \qquad (\text{LAISIN}, 2018)$$

2.8 Standard Basis

Suppose $\mathfrak{D}^\theta = F^m$ is the space of diagonal vectors and let the diagonal vector be denote e_i with $b_0(B) = 1$ in the i^{th} position and zeros elsewhere. Then, the m vectors e_i from a basis for F^m . That is every vector $X = (x_1, x_2, \dots, x_k)$ has the unique expression;

$$XE = x_1e_1 + x_2e_2 + \dots + x_me_m$$

as the linear combination of $E = (e_1,\ e_2,\ \dots,\ e_m)$ (LAISIN, 2018).

2.9 Lemma
There is no bishop's tour that visits every black cell on an 3 ≥ n board if n ≥ 7(Rouse Ball and Coxeter,1974; Gabriela, Sanchis and Hundley, 2004).

2.10 Lemma
Any bishop's tour that visits every black cell on an m × n board (m ≤ n and m ≥ 5) must begin or end on one of the cells on the outer boundary of the board. (Rouse Ball and Coxeter 1974; Gabriela, Sanchis and Hundley, 2004).

2.11 Lemma
A bishop's tour that visits all the black cells exists on a $m \times n$ board where m ≥ 1 and n ≥ 1, if and only if one also exists on an (m + 4) × (n + 4) board (Rouse Ball and Coxeter,1974; Gabriela, Sanchis and Hundley, 2004).

2.12 Theorem
The number of ways to arrange n bishops among m positions $(m \geq n)$ through an angle of $\theta = 45^0$ for movement on the board with forbidden positions is;

$$\mathfrak{B}(y, B)P_{(m,\ n)} = \sum_{k=0}^{n} (-1)^k b_k^{\theta}\, P_{(m-k,\ n-k)} \quad (\text{LAISIN}, 2018)$$

2.13 Theorem (n-disjoint sub-boards with movements through an angle of 45^0)
Suppose, B is an $n \times n$ board of darkened squares with bishops that move through a direction of an angle of $\theta = 45^0$ then, $\mathfrak{B}(x, B)$ for the disjoint sub-boards is;

$$\mathfrak{B}(x, B) = \sum_{i=0}^{n} \prod_{k=0}^{n} \mathcal{X}_{B_{j,K}}(x)^i b_i^{\theta}(B_j),\ \ j = 1,2, \dots n \quad (\text{LAISIN}, 2018)$$

3.0 Results

THEOREM 3.1
Suppose B is a two-dimensional chess board with maximum forbidden squares and let t non-attacking combination of rook and bishop movements generate a rook and bishop function, then, the generating function is;

$$r_0 b_0 + R(x, B)\mathfrak{B}(x, B) \geq \mathbb{Q}(x, B)$$

Proof
Considering 8×8 chess board of maximum forbidden squares. Then a combination of a rook and bishop movement on a two-dimensional board is as follows;

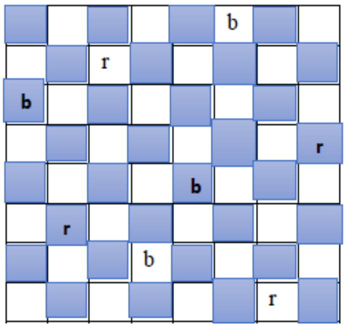


Fig 3.1

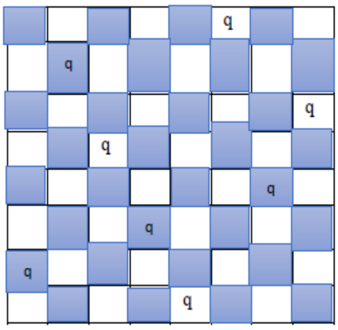


fig 3.2

Now, considering fig. 3.1, we have;

$$R(x,B) = \sum_{i=0}^{3} (x)^i r_i^{\theta}(B), \quad \theta = \begin{cases} 1, & if \quad 0^{\circ},\ 90^{\circ} \\ 0, & if\ otherwise \end{cases}$$

$$= r_0 + r_1 x + r_2 x^2 + r_3 x^3$$

$$\mathfrak{B}(x,B) = \sum_{i=1}^{4} (x)^i b_i^{\theta}(B), \quad \theta = \begin{cases} 1, & if\ 45^{\circ},\ 135^{\circ} \\ 0, & if\ otherwise \end{cases}$$

$$= b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4$$

The combined t non attacking rook and bishop movements is as follows;

$$R(x,B)\mathfrak{B}(x,B) = r_0 b_0 + (r_0 + r_1 x + r_2 x^2 + r_3 x^3)(b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4)$$

$$= r_0 b_0 + r_0(b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4) + r_1(b_1 x^2 + b_2 x^3 + b_3 x^4 + b_4 x^5)$$

$$+ r_2(b_1 x^3 + b_2 x^4 + b_3 x^5 + b_4 x^6) + r_3(b_1 x^4 + b_2 x^5 + b_3 x^6 + b_4 x^7)$$

$$= r_0 b_0 + r_0 b_1 x + (r_0 b_2 + r_1 b_1)x^2 + (r_0 b_3 + r_1 b_2 + r_2 b_1)x^3$$

$$+ (r_0 b_4 + r_1 b_3 + r_2 b_2 + r_3 b_1)x^4$$

$$+ (r_1 b_4 + r_2 b_3 + r_3 b_2)x^5 + (r_2 b_4 + r_3 b_3)x^6 + r_3 b_4 x^7$$

$$= r_0 b_0 + \sum_{i=0}^{3} \prod_{j=1}^{4} r_i^{\theta}(B) b_j^{\theta}(B) x^{i+j}, \quad \theta = \begin{cases} 1, & if\ 0^{\circ}, 45^{\circ},\ 90^{\circ}\ 135^{\circ} \\ 0, & if\ otherwise \end{cases}$$

$$\geq \sum_{i=0}^{7} q_i^{\theta}(B)x^i, \quad \theta = \begin{cases} 1, & if\ 0^{\circ}, 45^{\circ}, 90^{\circ}\ 135^{\circ} \\ 0, & if\ otherwise \end{cases}$$

Since, any 8×8 two-dimensional board can be increases in size by adding rows and columns respectively, then, it follows with the addition of a non-attacking rook followed by the addition of a non-attacking bishop for each pair of row and column added.

Now, it follows that an $t \times t$ two-dimensional board has $\frac{t}{2}$ non attacking rooks and $\frac{t}{2}$ non attacking bishops. Thus, the combination of the above with t non attacking rooks and bishops respectively, gives the following results for an $t \times t$ board;

$$R(x,B) = \sum_{i=0}^{t-1} (x)^i r_i^{\theta}(B), \quad \theta = \begin{cases} 1, & if\ \ 0^{\circ},\ 90^{\circ} \\ 0, & if\ otherwise \end{cases}$$

$$\mathfrak{B}(x,B) = \sum_{i=1}^{t} (x)^i b_i^{\theta}(B), \quad \theta = \begin{cases} 1, & if\ \ 45^{\circ},\ 135^{\circ} \\ 0, & if\ otherwise \end{cases}$$

where one is the angular direction for a rook or bishop movement on a two-dimensional board. Now considering fig 3.1 and fig 3.2 as an increased $t \times t$ board, then, we have;

$$= r_0 b_0 + \sum_{i=0}^{t-1} \prod_{j=1}^{t} r_i^{\theta}(B) b_j^{\theta}(B) x^{2i-1} \quad \geq \sum_{i=0}^{t} q_i^{\theta}(B)x^{i-1}, \ \ \theta = \begin{cases} 1, & if\ 0^{\circ}, 45^{\circ}, 90^{\circ}\ 135^{\circ} \\ 0, & if\ otherwise \end{cases}$$

$$r_0 b_0 \ + \ R(x,B)\mathfrak{B}(x,B) \geq \ \mathbb{Q}(x,B)$$

■

4.0 NUMERICAL APPLICATIONS

Example 4.0

A banana plantation with eight square kilometers is to be worked by a maximum pair of non-attacking controllers such that the first controller works diagonally, and the second works horizontally or vertically. If each plot is worked only by one controller. What percentage of the plot is worked by each controller? How many pairs of controllers can be given this assignment and in how many ways?

Solution

Let b = the first controller who works diagonally, and r = the second controller who works horizontally or vertically.

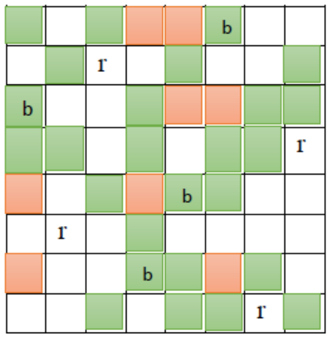


Fig 4.0

a) The first controller in green works a total of 40.63% of the plot while the second controller in white works 46.88% of the plot and 12.5% in red of the plot is yet to be assigned.

b) Four pairs of controllers give a maximum pair of non-attacking controllers such that the first works diagonally, and the second works horizontally or vertically.

c) $R(x,B)\mathfrak{B}(x,B)$

$$= r_0b_0 + \sum_{i=0}^{3}\prod_{j=1}^{4} r_i^{\theta}(B)b_j^{\theta}(B)x^{i+j}, \theta = \begin{cases} 1, if\ 0^{\circ}, 45^{\circ},\ 90^{\circ}\ 135^{\circ} \\ 0, \quad\quad if\ otherwise \end{cases}$$

$$R(x,B)\mathfrak{B}(x,B) = r_0b_0 + r_0b_1x + (r_0b_2 + r_1b_1)x^2 + (r_0b_3 + r_1b_2 + r_2b_1)x^3$$

$$+(r_0b_4 + r_1b_3 + r_2b_2 + r_3b_1)x^4 + (r_1b_4 + r_2b_3 + r_3b_2)x^5 + (r_2b_4 + r_3b_3)x^6 + r_3b_4x^7$$

$$= [1 + 4x + 15x^2 + 14x^3 + 14x^4 + 8x^5 + 3x^6 + x^7]$$

$$= 911\ ways$$

Example 4.1.

1. Determine the rook generating function for its movement on the forbidden space below.

		r_2	
	r_1		
r_0			

$$R(x,B_1) = r(x,B_1) + r_1(x,B_1)x + r_1(x,B_1)x^2$$

$$= 1 + 3x + x^2$$

2. Determine the bishop generating function for its movement on the forbidden space below.

b_2			
b_1		b_3	b_4
b_0			

$$\mathfrak{B}(x, B_1) = b_0(x, B_1) + b_1(x, B_1)x + b_2(B_1)x^2 + b_3(B_1)x^3 + b_4(B_1)x^4$$

$$= 1 + 5x + 6x^2 + x^3 + x^4 + x^5$$

3. Determine the rook generating function for its movement on the forbidden space below.

r_0	
	r_1

and

	r_1			
r_0				
		r_2		
				r_3

$$R(x, B_1) = r_0(x, B_1) + r_1(x, B_1)x$$

$$= 1 + 2x$$

$$R(x, B_2) = r_0(B_2) + r_1(B_2)x + r_1(B_2)x^2 + r_1(B_2)x^3$$

$$= 1 + 4x + 3x^2 + 2x^3$$

$$R(x, B) = 1 + 6x + 11x^2 + 8x^3 + 4x^4$$

$$R(x, B_1)R(x, B_2) = 1 + 8x + 12x^2 + 30x^3 + 20x^4 + 8x^5$$

4. Determine the rook and bishop generating function for its movement on the forbidden space below.

and

		r_2	
	r_1		
r_0			

b_2			b_3
b_1			b_5
b_0			b_4

$$R(x, B_1) = r_0(B_1) + r_1(B_1)x + r_2(B_1)x^2$$

$$= 1 + 2x + x^2$$

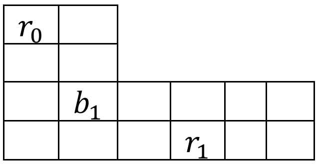
$$\mathfrak{B}(x, B_2) = b_0(x, B_2) + b_1(x, B_2)x + b_2(B_2)x^2 + b_3(B_2)x^3 + b_4(B_2)x^4 + b_5(B_2)x^4 +$$

$$b_6(B_2)x^4$$

$$\mathfrak{B}(x, B_2) = 1 + 7x + 15x^2 + 10x^3 + x^4 + 3x^5 + 2x^6$$

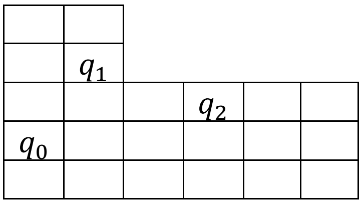
$$R(x, B_1)\mathfrak{B}(x, B_2) = 1 + 9x + 30x^2 + 47x^3 + 36x^4 + 15x^5 + 9x^6 + 7x^7 + 2x^8$$

5. Determine the rook and bishop generating function for its movement on the forbidden space below.



$$R(x, B)\mathfrak{B}(x, B) = r_0 b_0(B) + r_0 b_1(B) + r_1 b_0(B) + r_1 b_1(B)$$
$$= 1 + 2x + x^2$$

6. Determine the queen generating function for its movement on the forbidden space below.



$$Q(x, B) = q_0(B_1) + q_1(B_1)x + q_2(B_1)x^2$$
$$= 1 + 2x + x^2$$

Conclusion

The combination of polynomials generated by rook and bishop movements on a board with forbidden space for non-attacking rook and bishop is interesting for two-dimensional cases. We realized our objectives by showing that, the combined rook and bishop movement generate a generating function that is better on computational result to that of the queen generating function on a chessboard with forbidden space. Furthermore, we applied these techniques used in the construction of rook and bishop generating functions to solve problems for different cases on disjoined sub-boards.

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