OPTIMALITY CONDITIONS AND THE TARGET SET OF FUNCTIONAL DIFFERENTIAL SYSTEMS OF THE SOBOLEV TYPE IN BANACH SPACES WITH DISTRIBUTED DELAYS IN THE CONTROL

by

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ABSTRACT
In this paper, functional differential systems of Sobolev-type in Banach spaces with distributed delays in the control are presented for controllability analysis. Necessary and sufficient conditions for the existence of the optimal control of the systems were established. The form of the optimal control was obtained and the uniqueness of the optimal control of the system was established. Uses were made of the Signum function and some of the controllability standards to establish the results. We have also established that the target set of the system use to be on the boundary of the attainable set / reachable set once the optimal is applied.

Key Words: Functional Differential System, Sobolev-type, Banach space, Signum function, Controllability Standard, Set Functions.

1. INTRODUCTION

Controllability is one of the fundamental concepts in mathematical control theory. It is qualitative property of dynamical control systems and is of particular importance to the control theorist. Systematic study of controllability started over the years at the beginning of sixties when the theory of controllability based on the description in the form of state space for both time-varying and time-invariant linear control system was carried out (Oraekie 2016). Controllability generally means that, it is possible to steer a dynamical control system from an initial state to a final state using the set of admissible controls (Oraekie 2012).

In recent past, various controllability problems for different types of linear and nonlinear systems have been considered. In (Gyori and Wu, 1991), a simplified model for compartmental systems with pipes is represented by neutral voltera integro-differential equation. (Onwuatu, 1993) derived a set of sufficient conditions for the controllability of nonlinear neutral systems through the fixed point method. (Balachandran and Dauer, 1998)investigated the relative controllability of nonlinear systems with distributed delays in control. They derived sufficient conditions for the relative controllability of nonlinear neutral Volterra Integro-differential systems with distributed delays in the control variables. The results were obtained by using Schauder's fixed point theorem. Success in life revolves around the setting of targets and the achievement of same. Controllability presumes a predetermined target and effort is geared toward the selection of initial point and control energy that will steer the state of system at the initial point to a terminal point (desired) in finite time (Oraekie, 2012).The set of such initial point is called the core oftarget denoted by Co (H), while the set of terminal points form the target set, denoted by H**.** The controllability of functional differential systems of Sobolev type in banach Spaces had been established by (Balachandran and Dauer, 1998). According to them, the problem of controllability of nonlinear systems represented by ordinary differential equations of infinite dimensional spaces had been extensively studied.

Our aim/reason, therefore, is to present an organized treatment of control theory that could be complete within the limitations set by the restrictions of deterministic problems identifiable in terms of functional differential equations. There are two kinds of functional differential equations, namely:

(a) The Retarded functional differential equation given as

$$\dot{x} = f(t, x_t);\ \mathrm{x(t_0)} = \emptyset = x_{t_0} \qquad (1.0.1)$$

Where$\emptyset$is the initial function defined in the delay interval $[-h, 0], h > 0$.

(b) The neutral functional Differential equation given as

$$\frac{d}{dt}[D\ (t, x_t)] = f\ (t, x_t);\ \mathrm{x(t_0)} = \mathrm{x_{t_0}} \qquad (1.0.2)$$

where D is a bounded linear operator.

The differential equations which include the present as well as the past state of any physical system is called a Delay Differential Equation or Functional Differential Equations (Oraekie 2012). The retarded functional Differential Equations (RFDE) are characterized by delays in the state of the system. A typical example is the system below:

$$\frac{dx(t)}{dt} = x(t-h), h > 0 \qquad (1.0.3)$$

On the other hand, Neutral Functional Differential Equations (NFDE) are those that have delays in the state as well as in the derivatives.

An example is the system below:

$$\frac{d}{dt}[x(t) - c(x(t-h)] = bx(t-h) \Rightarrow \dot{x}(t) - c(\dot{\mathrm{x}} - h) = bx(t-h) (1.0.4)$$

Our specific objective is to obtain necessary and sufficient conditions for: controllability of the system, existence of optimal control, form and the position of target set when an optimal control is in work.

2. PRELIMINARIES

2.1 DEFINITIONS

DEFINITION 2.1.1: (CONTROLLABILITY)

The linear system (1.2.3) is said to be controllable if and only if for any initial state $x_1\ at\ time\ t_1$ there exists steering function $u(.)$ which steers the system from $x_1\ at\ t_1$ to $x_2\ at\ time\ t_2$ in finite time.

That is for any initialstatex_1,initialtimet_1 given, there exists timet_2 and $u(.)$ such that

$$\boldsymbol{\psi(x_1, t_1, u(.), t_2) = x_2}$$

DEFINITION 2.1.2: (NULL-CONTROLLABILITY)

The linear system (2.3.1)is null-controllable if and only if for any initial state $\boldsymbol{x_1}\ at\ \boldsymbol{t_1}$ there exists steering function $\boldsymbol{u}(.)$ which steers the system from $\boldsymbol{x_1}\ at$ time$\boldsymbol{t_1}$ to $\boldsymbol{x_2} = \boldsymbol{0}at\ \boldsymbol{t_2}$ in finite time. That is for any $\boldsymbol{x_1, t_1}$ given, there exists $\boldsymbol{t_2}$ and $\boldsymbol{u}(.)$ such that $\boldsymbol{\psi(x_1, t_1, u(.), t_2) = 0}$.

We state, without proof, the following very important result that provides criteria for determining the null-controllability for the system (2.3.1).

DEFINITION 2.1.3: (REACHABLE SET)

$$Consider\ the\ system\ (2.3.1)\ given\ as$$

$$\dot{x} = A(t)\dot{x} + Bu(t) \qquad (2.3.1)$$

$$Let\ the\ solution\ be\ x(t) such\ that$$

$$\mathrm{x(t) = X(t)X^{-1}(t_0)x_0 + X(t)\int_{t_0}^{t_1} X^{-1}\,(s)B(s)u(s)ds}$$

where$X$(t) is a fundamental matrix and $X(t_0) = X(0) = I$.

We define the reachable set as

$$R\{(t_0, t_1) = \int_{t_0}^{t_1} X^{-1}\,(s)B(s)u(s)ds: u\epsilon U\}$$

where U is the set of admissible controls.

DEFINITION 2.1.4: (ATTAINABLE SET)

Attainable set is the set of all possible solutions of a given control system. In the case of the system (2.3.1), for instance, it is given as

$$A\{(t_0, t_1) = \{x(t) = X(t)X^{-1}(t_0)x_0 + X(t)\int_{t_0}^{t_1} X^{-1}\,(s)B(s)u(s)ds: u\epsilon \mathrm{U}\},$$

$Evidently,\ R\{(t_0, t_1)$is a translation of attainable set through the origin x_0, that is

$$A\{(t_0, t_1) = \{x(t) = X(t)X^{-1}(t_0)x_0 + X(t)\int_{t_0}^{t_1} X^{-1}\,(s)B(s)u(s)ds: u\epsilon \mathrm{U}\},$$

$$= X(t)X^{-1}(t_0)x_0 + \int_{t_0}^{t_1} X^{-1}\,(s)B(s)u(s)ds: u\epsilon U]$$

$= x_0 + \int_{t_0}^{t_1} X^{-1}\,(s)B(s)u(s)ds: u\epsilon U \qquad = x_0 \ + \ R(t_0, t_1),$

since X(t) is a fundamental matrix and fundamental matrices are invertible .

DEFINITION 2.1.5: (PROPERNESS)

The system (2.3.1), given as

$\dot{\mathrm{x}} = \mathrm{A(t)}x(t) + B(t)u(t)$ is proper on the interval $[t_0, t_1]$ if and only if

$\mathrm{C^T X^{-1}(t)B(t)} = 0$, i.e on $[t_0, t_1]$, implies that $C = 0$.

Here, the set function

$\mathrm{g(t) = C^T X^{-1}(t)B(t)}$is called the controllability index.

2.2 DESCRIPTION OF SYSTEM AND VARIATION OF CONSTANT FORMULA

Consider the Sobolev-Type functional differential systems with distributed delays in the control of the form

$$(\mathrm{Q}x(\mathrm{t}))^1 \ + \ \mathrm{A}\,x(t) \ = \ f(t_1, x_t) \ + \ \int_{-h}^{0} d_\theta\, H(t,\theta)u(t+\theta) (2.1)$$

$x(t) \ = \ x_t = \phi(t); \quad -h \leq \ t \ \leq 0, or\ t\epsilon[-h, 0]$

Where the state $x(.)$ in the solution and takes values in the Banach space X and the control function u(.) is given in L^2 (J,U) a Banach space of admissible control functions with U as a Banach space since U, the constraint controls set is closed and bounded subset of $L_2(t_0, t_1], X)$. Here $x\epsilon E^n$ and u in an admissible square integrable m-dimensional vector function, with $|u_j| \leq 1, j = 1,2, \dots, m. H(t, \theta)$ *is an nxm matrix continuous at t and of bounded variation in* θ on $[-h, 0]$; $h > 0$ for each $t\epsilon[0, t_1]$; $t_1 > 0$. The nxn matrices: A and B are continuous in their arguments. The n-vector function g is absolutely continuous. The integral is in the Lesbesgue – stieltjes sense and is denoted by the symbol $d\theta$.

The control space will be the Lebesgue space of square integrable functions, $L_2([t_0, t_1], E^n)$

Let $h > 0$, be given, for a function $u: [-\mathrm{h}, \mathrm{t}_1] \to E^n$ and $t\epsilon[0, t_1]$, we use the symbol $u(t)$ to denote the function defined on the delay interval $[-h, 0]$ by.

$$\mathrm{u_t(s) = u(t+s)\ for\ s\epsilon[-h, 0]}$$

2.2.1VARIATION OF CONSTANT FORMULA

By integrating system, we obtain an expression for solution as in (Balachandran 1992, Balachandran and Dauer 1989) and (Balachandran, K and Sakthivel, R. 1999) as contained in (Oraekie 2012)

$$x(t) = Q^{-1}T(t)Qx_0 + \int_{t_0}^{t} Q^{-1}T(t-s)f(s, x_s)ds$$
$$+ \int_{t_0}^{t} Q^{-1}T(t-s) \int_{-h}^{0} d_\theta\, H(t,\theta)u(t+\theta) \quad (2.1)$$
$$x(t) = \phi(t);\ -h \leq t \leq 0 \quad \forall\, t$$

A careful observation of the solution of the system shows that the values of the control *u(t)* for $t\epsilon[-h, t_1]$ enter the definition of complete state thereby creating the need for an explicit variation of constant formula. The control in the last term of the formula (1.1), therefore, has to be separated in the intervals $[-\mathrm{h}, 0]$and $[0, t_1]$,

To achieve this, that term has to be transformed by applying the method of Klamka (1980). Finally, we interchange the order of integration using the unsymmetric fubinis theorem to have.

$$x(t) = Q^{-1}T(t)Qx_0 + \int_{t_0}^{t} Q^{-1}T(t-s)\, f(s, xs)\, ds$$

$$+ \int_{-h}^{0} dH_\theta \int_{0+\theta}^{t+\theta} Q^{-1}T(t-s)\, H(s-\theta, \theta)u(s)ds \quad (2.3)$$

simplifying (2.3), we have

$$x(t) = Q^{-1}T(t)Qx_0 + \int_{t_0}^{t} Q^{-1}T(t-s)\, f(s, xs)\, ds$$
$$+ \int_{-h}^{0} dH_\theta \int_{\theta}^{0} Q^{-1}T(t-s)H(s-\theta, \theta)u_0(s)\, ds$$
$$+ \int_{-h}^{0} dH_\theta \int_{0}^{t+\theta} Q^{-1}T(t-s)H(s-\theta, \theta)u(s)ds \quad (2.4)$$

Using again, the unsymmetric fubinis theorem on the charge of order of integration and incorporating H* as defined below:

$$H^*(s,\theta) = \begin{cases} H\,(s,\theta), for\ s\leq t \\ 0 \quad for\ s\geq t \end{cases} \qquad (2.5)$$

The formula (2.4) becomes

$$x(t) = Q^{-1}T(t)Qx_0 + \int_{t_0}^{t} Q^{-1}T(t-s)\,f(s,x_s)\,ds$$

$$+dH_\theta \int_{\theta}^{0} Q^{-1}T(t-s\,)H\,(s-\,\theta,\theta)u_0(s)\,ds$$

$$+\int_{-h}^{0} dH_\theta \int_{0}^{t+\theta} Q^{-1}T(t-s)H\,(s-\,\theta,\theta)u\,(s)\,ds$$

$$\Rightarrow x(t) = Q^{-1}T(t)Qx_0 + \int_{t_0}^{t} Q^{-1}T\,(t-s)f(s,xs)\,ds$$

$$+\int_{-h}^{0} dH_\theta \int_{\theta}^{0} Q^{-1}T(t-s\,)H\,(s-\,\theta,\theta)u_0(s)\,ds$$

$$+\int_{0}^{t}\{\int_{-h}^{0} Q^{-1}\,T(t-s)H\,(s-\,\theta,\theta)u\,(s)\,\}ds \qquad (2.6)$$

The integration is still in the Lebesgue Stieltjes sense in the variable θ in H.
For brevity, let

$$\beta(\mathrm{t}) = Q^{-1}T\,(t)Qx_0 + \int_{t_0}^{t} Q^{-1}T(t-s)f(s,xs)ds \qquad (2.7)$$

$$\mu(\mathrm{t}) = \int_{-h}^{0} dH_\theta \int_{\theta}^{0} Q^{-1}\,T(t-s)H(S-\,\theta,\theta)U0(s)ds \qquad (2.8)$$

$$\mathrm{Z}\,(\mathrm{t,s}) = \int_{-\mathrm{h}}^{0} \mathrm{Q}^{-1}T(t-s)d\theta\mathrm{H}^*(s-\,\theta,\theta)u\,(s)\,\} \qquad (2.9)$$

Substituting (2.7), (2.8) and (2.9) into (2.6) we have a précised variation of formula for the system (1.1.1) as

$$x(t_1,x_0,u) = \beta\,(\mathrm{t}) + \mu\,(\mathrm{t}) + \int_0^t \mathrm{Ƶ}(t,s)u(s)ds \qquad (3.0)$$

2.2.2 BASIC SET FUNCTIONS

Application will be made of the following basic set functions and properties upon which our study hinges.

i. $Reachable\ Set\ R(t_0, t_1) = \{\int_0^t \int_{-h}^0 Q^{-1}T(t-s)d_\theta H^*(s-\theta,\theta)u(s)\ ds:\ u\epsilon U\}$

ii. $Attainable\ Set\ A(t_0, t_1) = \{x(t_1, x_0, u): u\epsilon U\}$

iii. Controllability map or grammian and $= Z(t,s)Z^T(t,s)ds$

iv. Controllability index $= Q^{-1}T(t-s)d_\theta H^*(s-\theta,\theta)$

v. Target Set

The target set $G(t_1, t_0)$of the system (1.1) is given as:
$G(t_1, t_0) = \{x(t_1, x_0, u): t_1 \geq t > t_0 = 0$, for fixed$tandu\epsilon U\}$

vi. Relative Controllability

System (2.1) is said to be relatively controllable on the interval $[t_0, t_1]$ if and only if

$$A(t_0, t_1) \cap G(t_1, t_0) \neq \emptyset;\ t_1 > t_0 = 0$$

vii. Properness

System (2.1) is said to be proper in E^n on the interval $[t_0, t_1]$ iff $R(t_1, t_0) = E^n$
i.e.

$$C^T \int_{t_0}^{t_1} Q^{-1}T(t-s)d_\theta H^*(s-\theta,\theta)u(s)ds = 0 \Rightarrow C = 0, C \in E^n$$

3. MAIN RESULTS

The optimal control problem can best be understood in the context of a game of pursuit (Oraekie, 2016).The emphasis here is the search for a control energy function that can steer the state of the system of our interest to the target set $G(t_1, t_0\ )$which can be a moving point function or a compact set function in minimum time. In other words, the optimal control problem in stated as follows:

If

$$t^* = infimum\{t: A(t_1, t_0\ )\ \cap G(t_1, t_0\ ) \neq \emptyset\}$$

For $t\epsilon[t_1, t_0];\ \ t_1 > t_0 = 0.$

That is, if t* is the minimum of all the times such that the system of our interest is relatively controllable, does there exists an admissible control u* such that the solution of the system with this admissible control be steered into target?

The theorem that follows answers in part the questions.

Theorem 3.1: (Existence Conditions)

Consider the system (1.1) as a differential of pursuit.

$$(Qx(t))^1 + Ax(t) = f(t, x_t) + \int_{-h}^0 d_\theta H(t,\theta)u(t+\theta)$$
$$x(t) = Q(t); t\epsilon[-h, 0];\ \ h > 0$$

with its standing assumptions,

Suppose A(t_1, t_0) and G(t_1, t_0) are compact set functions, then there exists an admissible control such that the state of the weapon for the pursuit of the target satisfies system (1.1) if and only if.

$$A(t_1,t_0)\ \cap G(t_1,t_0)\ \neq \emptyset \quad for\ t\epsilon[t_0,t_1]$$

Proof:

Suppose that the state z(t) of the weapon for pursuit of the target satisfies system (1.1) on the time interval $[t_0,t_1]$, $then$

$$z(t)\ z(t)\epsilon G(t_1,t_0) for\ t\epsilon(t_0,t_1).$$

$We\ need\ to\ show\ that\ there\ exists\ a\ solution\ x(t,u)\epsilon A(t_1,t_0) for\ t\epsilon(t_0,t_1)\ such\ that$ $z(t)=x(t,u)$, for some uϵU.

Let $\{u^n\}$ be a sequence of point in U. Since the constraint control set U is compact, then the sequence $\{u^n\}$ has a limit u as n tends to infinity.

Now, $x(t_1,x_{0,}u)\ \epsilon$A (t_1,t_0), for t$\in$(t_0, t_1) and from system (2.6), we have

$$x(t)=x(t,x_0,u^n)=Q^{-1}T(t)Qx_0+\int_{t_0}^{t_1} Q^{-1}T(t-s)f(s,x_s)ds$$
$$+\int_{-h}^{0} d_\theta\, H\int_{\theta}^{0} Q^{-1}T(t-s)H(s-\theta,\theta)U_0^n\,(s)ds$$
$$+\int_{t_0}^{t_1}\{\int_{-h}^{0} Q^{-1}\,T(t-s)d_\theta H^*(s-\theta,\theta)u^n(s)ds\} \quad (3.1)$$

Taking limits on both sides of system (3.1) above as n tends to infinity, we have

$$\lim_{n\to\infty} x\big(t,x_{0,}u^n\big)=Q^{-1}T(t)Qx_0+\int_{t_0}^{t_1} Q^{-1}T(t-s)f(s,x_s)ds$$
$$+\int_{-h}^{0} d_\theta\, H\int_{\theta}^{0} Q^{-1}T(t-s)H(s-\theta,\theta)\lim_{n\to\infty}U_0^n(s)ds$$
$$+\int_{t_0}^{t_1}\{\int_{-h}^{0} Q^{-1}\,T(t-s)d_\theta H^*(s-\theta,\theta)\lim_{n\to\infty}u^n(s)\}ds$$
$$=x(t,x_0,u^n)=Q^{-1}T(t)Qx_0+\int_{t_0}^{t_1} Q^{-1}T(t-s)f(s,x_s)ds$$
$$+\int_{-h}^{0} d_\theta\, H\int_{\theta}^{0} Q^{-1}T(t-s)H(s-\theta,\theta)u_0\,(s)ds$$
$$+\int_{t_0}^{t_1}\left\{\int_{-h}^{0} Q^{-1}\,T(t-s)d_\theta H^*(s-\theta,\theta)u(s)\right\}ds$$

$=x(t,x_0,u)\in A(t_1,t_0)$ since $A(t_1,t_0)$ is compact

$Thus, there\ exists\ a\ control\ u\in U\ such\ that\ x(t,x_0,u)=z(t) for\ t>t_0\ and$ $t\in[t_0,t_1]$.

$Since\ z(t)\in G(t_1,t_0)\ and\ also\ in\ A(t_1,t_0), it\ follows\ that$

$$A(t_1,t_0) \cap G(t_1,t_0) \neq \emptyset \qquad for\ t \in (t_0,t_1).$$

Conversely,
Suppose that the intersection condition holds (i.e. the system in relatively controllable)
i.e. $A(t_1,t_0) \cap G(t_1,t_0) \neq \emptyset \qquad for\ t \in (t_0,t_1)$.
Then, there exists $z(t) \in A(t_1,t_0)$such that $z(t) \in G(t_1,t_0)$. This implies that
$z(t) = x(t_1,x_0,u)$ and hence establishes that the state of the weapon of pursuit of the target satisfies system (1.1)

Remark 3.1
The above stated and proved theorem (Theorem 2.1) implies that, in any game of pursuit described by functional differential system of Sobolev–Type in Banach spaces with distributed delays in the control, it is always possible to obtain a control energy function $u(.)$ to steer the systems state $x(.)$ to the target in finite time. The next theorem (theorem 3.2) is therefore, a consequence of this understanding and provides us with sufficient conditions for the existence of a control energy function that is capable of steering the state of the system (1.1) to target set in minimum time.

Theorem 3.2: (Sufficient condition for the Existence of an Optimal Control)
Consider the system (1.1)

$$(Qx(t))^1 + Ax(t) = f(t,x_t) + \int_{-h}^{0} d_\theta\, H(t,\theta)u(t+\theta)$$

$x(t) = \emptyset(t); \qquad t \in [-h,0]$
With its basic assumptions, suppose that system (1.1) is relatively controllable on the finite interval $[t_0,t_1]$, then there exists an optimal control.
Proof
By the controllability of the system (1.1), the intersection condition holds:
i.e $A(t_1,t_0) \cap G(t_1,t_0) \neq \emptyset \qquad for\ t \in (t_0,t_1)$.
Hence, $x(t,x_0,u) \in A(t_1,t_0)$. Also $x(t,x_0,u) \in G(t_1,t_0)$
so, put $z(t) = x(t,x_0,u)$.
Recall that the attainable set $A(t_1,t_0)$ is a translation of the reachable set $R(t_1,t_0)$through the origin η i.e $A(t_1,t_0) = \eta + R(t_1,t_0)$
where η is given as in system (2.6) to be

$$\eta = Q^{-1}T(t)Qx_0 + \int_{t_0}^{t_1} Q^{-1}T(t-s)f(s,x_s)ds$$
$$+ \int_{-h}^{0} d_\theta\, H \int_{\theta}^{0} Q^{-1}T(t-s)H(s-\theta,\theta)u_0\,(s)ds$$

Then, $\quad A(t_1,t_0) = Q^{-1}T(t)Qx_0 + \int_{t_0}^{t_1} Q^{-1}T(t-s)f(s,x_s)ds$

$$+ \int_{-h}^{0} d_\theta\, H \left\{ \int_{\theta}^{0} Q^{-1}T(t-s)H(s-\theta,\theta)u_0\,(s)ds \right\} + R(t_1,t_0)$$

It follows that z (t) $\in$ R (t,t_0) for t$\in (t_1,t_0)$; $t_1 > t_0$ and can be defined as:

$$z(t) = \int_{t_0}^{t_1} \{ \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta) u(s) \} ds$$

$|u_j| \leq 1$, for very $u_j \in U \subseteq L_2(\text{J}, E^m)$, J= $[t_0, t_1]$, $(\text{j} = 1,2,3, \dots \dots, \text{m})$.
Let t* = infimum{t :z (t) ∈R $(\mathrm{t}_1, \mathrm{t}_0)$ for t∈$[\mathrm{t}_1, \mathrm{t}_0]$} .
Let $t_0 \leq t_n \leq t_1$, and there is a sequence of time $\{\mathrm{t_n}\}$ and the corresponding control u^n such that $\{u^n\} \subset U$ with the sequence ($\mathrm{t_n}$) converging to t* - the minimum time.
Let $z(t_n) = y(t_n, u^n) \in \mathrm{R}(\mathrm{t}_1, \mathrm{t}_0)$.
Also,

$$\begin{aligned} |z(t^*) - y(t^*, u^n)| &\leq |z(t^*) - z(t_n) + z(t_n) - y(t^*, u^n)| \\ &\leq |z(t^*) - z(t_n)| + |z(t_n) - y(t^*, u^n)| \\ &\leq |z(t^*) - z(t_n)| + |z(t_n, u^n) - y(t^*, u^n)| \\ &\leq |z(t^*) - z(t_n)| + \int_{t^*}^{t_n} \|y(s)\| \, ds \end{aligned}$$

By the continuity of z(t), which follows the continuity of reachable set $\mathrm{R}(\mathrm{t}_1, \mathrm{t}_0)$ as a continuous set function and the integrability of $\|y(s)\|$, it follows that

$$y(t^*, u^n) \rightarrow z(t^*) \; as \; n \rightarrow \infty, where$$
$$z(t^*) = y(t^*, u^*) \epsilon \mathrm{R}(\mathrm{t}_1, \mathrm{t}_0).$$

For some u*ϵU and by the definition of t*; u* is am optimal control.
This establishes the existence of an optimal control for the Functional Differential Systems of Sobolev-Type in Banach Spaces with Distributed Delays in the Control.

3.2 THE FORM OF OPTIMAL CONTROL AND THE LOCATION OF THE TARGET SET.

Let us derive the form of the optimal control for the Functional Differential Systems of Sobolev-Type in Banach Spaces with Distributed Delays in the Control and express some using the definition of the Signum function.

Theorem 3.3: (Form of Optimal Control)
Consider the system (1.1)

$$(Qx(t))^1 + Ax(t) = f(t, x_t) + \int_{-h}^{0} d_\theta \, H(t,\theta) u(t+\theta)$$

$x(t) = \emptyset(t); \qquad t \in [-h, 0]$; h > 0
With its basic assumption, then:
U* is the optimal control energy function for the system (1.1) if and only it u* is of the form:

$$\mathrm{u}^*(\mathrm{t}) = \ \mathrm{sgn}\ C^T \int_{-h}^{0} Q^{-1} T\,(\mathrm{t}-\mathrm{s}) d_\theta \mathrm{H}^*(\mathrm{s}-\theta,\theta);\ Where\ C \in E^n$$

Proof:
Suppose, u* is the optimal control energy function for system (1.1), then it maximizes the rate of change of $\mathrm{y}(\mathrm{t}, \mathrm{u}\,)$, where

$$y(t, u) = \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta) u(t)$$

for u∈U in the direction of C.

Since u(t) are admissible controls, that is, they are constrained to lie in a unit sphere, where we have

$$C^T \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta) u(t) \leq \left| C^T \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta) u(t) \right|$$

$$\leq \left| C^T \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta) \right|$$

$$\leq C^T \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta)\ Sgn\ C^T \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta) 2.3.1$$

This inequality follows from the fact that for any non-zero number $a,\ b \leq b\ Sgn\ b$.
Hence, defining

$$u^* = Sgn \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta)$$

In inequality (2.3.1), we have

$$C^T \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta) u(t) \leq C^T \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta) u^*(t)$$

This shows that the control that maximizes $y(t,u) \in R(t_{1,}t_0)$ is of the form

$$u^* = Sgn\ C^T \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta)$$

Conversely,
Let u $\in$ U be given as

$$U = Sgn\ C^T \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta)$$

Then, for the control u$\in$ U

$$C^T \int_{t_0}^{t_1} \{ \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta) \} u(s) ds$$

$$\leq \int_{t_0}^{t_1} C^T \{ \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta) \} ds$$

$$\leq Sgn \int_{t_0}^{t_1} \{ C^T \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta) \} ds$$

$$\leq \int_{t_0}^{t_1} C^T \{ \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta) \}. ds$$

(Since for $b \neq 0,\ b\ Sgn\ b > 0$)

$$\leq \int_{t_0}^{t_1} C^T\{ \int_{-h}^{0} Q^{-1}\, T(t-s) d_\theta H^*(s-\theta,\theta)\} u^*(s) ds$$

This shows that u* maximizes

$$C^T \int_{t_0}^{t_1} \{ \int_{-h}^{0} Q^{-1}\, T(t-s) d_\theta H^*(s-\theta,\theta)\} u(s) ds$$

over all admissible control u∈U.

Hence u* is an optimal control for system (1.1). This completes the proof.

Remark 3.3

It is evident from theorem 3.3 that if u* is the optimal control, then the target is on the boundary of the reachable set. To see this let

$$y^* = y(t^*, u^*) = \int_{t_0}^{t_1} \{ \int_{-h}^{0} Q^{-1}\, T(t-s) d_\theta H^*(s-\theta,\theta)\} u^*(s) ds$$

for $t \in [t_0, t_1]$

Then, from the result of theorem 3.3

$C^T y \leq C^T y^* \Rightarrow C^T(y-y^*) \leq 0$ for $y \in R[t_1, t_0]$

Since the reachable set R(t_1, t_0) is a closed, convex subset of E^n, there in a support plane π of R (t_1, t_0) through C with $C \neq 0$, an outward nomnal to π at y* and hence y* is in the boundary of the reachable set. This realization is formalized in the next theorem.

Theorem 3.4: (Target set on boundary of A(t_1, t_0))

Let u* be the optimal control for system (1.1) with t* the minimum time, then the target z (t*) = x (t*, x_0, u*) is on the boundary of the attainable set, A(t_1, t_0).

i.e. z(t*) ∈ δA (t_1, t_0)

(where δ symbolizes boundary) for t,t* ∈ (t_1, t_0).

Proof

Suppose that u* is an optimal control, then

$$x(t^*, u^*) = [\eta + y^*] \in R(t^*, t_0)$$

Therefore, $x(t^*, u^*) \in$ A(t_1, t_0)

Now, suppose for contradiction x(t*, u*) is not on the boundary, then x(t*, u*) is in the interior of A(t_1, t_0); t* ∈ $[t_1, t_0]$.

Therefore, there is a ball B (x(t*, u*), r) contred at x(t*, u*) radius r in A(t*, t_0) because A(t_1, t_0) is a continuous set function of t, we can preserve the above inclusion for t near t*.

If we reduce the size of the ball B (x(t*, u*), r); that is, if there is an $\varepsilon > 0$ such that

B(x (t*, u*), $r/2$)⊂A (t, t_0) for t*- ε ≤ t ≤ t* and t, t* ∈ (t_0, t_1)

Thus,

x(t*, u*) ∈ A (t, t_0) for t*- ε ≤ t ≤ t*.

This, of course contradicts the optimality of t* and u* as the optimal time and optimal control. This contradiction, however, proves that the target z (t*) is on the boundary of the attainable set A (t*, t_0) and hence on the boundary of the reachable set. R(t*, t_0).

Remark 3.4

This theorem 2.4 is the basis of Pontrygin's maximum principle as contained in (Oraekie, 2016). There are other fascinating properties that emanate from the convexity property of the reachable set. In (Chukwu, 1992), it is stated that if the reachable set of a system is strictly convex, then, the system is said to be normal and optimal control of such a system is said to be Bang-bang. By the Bang-bang principle any reachable set that can be reached by an admissible control can be reached by Bang - bang control.

Theorem 3.5: (Uniqueness of Optimal Control)

Consider the system (1.1)

$$(Qx(t))^1 + Ax(t) = f(t, x_t) + \int_{-h}^{0} d_\theta\, H(t,\theta)u(t+\theta)$$

$x(t) = \emptyset(t);$

with its standing hypothesis.

Suppose u* is the optimal control, then it is unique.

Proof

Here, new a method of approach is derived for the proof of $t \in [-h, 0]$; h > 0 the existence of optimal system. Now, let u* and v* be optimal control, for system (1.1), them both u* and v* maximizes

$$C^T\{\int_{-h}^{0} Q^{-1}\, T(t-s)d_\theta H^*(s-\theta,\theta)\}$$

for $t \in [t_0, t_1]$; $t_1 > t_0$

over all admissible control u∈U, and so we have the inequality with u* as the optimal control

$$C^T \int_{t_0}^{t^*} \{ \int_{-h}^{0} Q^{-1}\, T(t-s)d_\theta H^*(s-\theta,\theta)\}u(s)ds$$

$$\leq C^T \int_{t_0}^{t^*} \{ \int_{-h}^{0} Q^{-1}\, T(t-s)d_\theta H^*(s-\theta,\theta)\}u^*(s)ds \qquad 3.5.1$$

Also, using V*, as optimal control, we have the inequality

$$C^T \int_{t_0}^{t^*} \{ \int_{-h}^{0} Q^{-1}\, T(t-s)d_\theta H^*(s-\theta,\theta)\}v(s)ds$$

$$\leq C^T \int_{t_0}^{t^*} \{ \int_{-h}^{0} Q^{-1}\, T(t-s)d_\theta H^*(s-\theta,\theta)\}v^*(s)ds \qquad 3.5.2$$

Taking maximum of u, over $[-1, 1]$, the range of definition of u* in (3.5.1) and (3.5.2), we have the equation.

$$C^T \int_{t_0}^{t} \{ \int_{-h}^{0} Q^{-1}\, T(t-s)d_\theta H^*(s-\theta,\theta)\} \max_{-1\leq s\leq 1} |u(s)|ds$$

$$= C^T \int_{t_0}^{t^*} \{ \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta)\} u^*(s) ds \qquad 3.5.3$$

for $u, u^* \in U$

Also,

$$C^T \int_{t_0}^{t} \{ \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta)\} \max_{-1\le s\le 1} |u(s)| ds$$

$$= C^T \int_{t_0}^{t^*} \{ \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta)\} v^*(s) ds \qquad 35.4$$

for $u, v^* \in U,\ v^*$ being optimal

Subtracting equation (3.5.3) from equation (3.5.4), we have

$$0 = C^T \int_{t_0}^{t^*} \{ \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta)\} \{v^*(s) - u^*(s)\} ds$$

$$\Rightarrow 0 = \{ \int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta)\} \{v^*(s) - u^*(s)\}$$

But $\int_{-h}^{0} Q^{-1} T(t-s) d_\theta H^*(s-\theta,\theta) \neq 0$

Therefore, $[v^*(s) - u^*(s)]$

$$\Rightarrow v^*(s) - u^*(s) = 0$$

$$\Rightarrow v^*(s) = u^*(s)$$

This establishes the uniqueness of the optimal control for the system (1.1).

CONCLUSION

In this paper, we have established the existence of optimal control, form and uniqueness of the optimal control. We used variation of optimal control fsormula and established the mild solution of the system (1.1). From this mild solution, we extracted the set functions upon which our studies hinged. New approach for the establishment of the existence of the optimal control was cultivated for Sobolev-type functional differential system in banach spaces with distributed delays in the control. Thus, we have extended the concept of optimal control of Sobolev-type functional differential system in banach spaces with one-point time delay in the control to distributed delays in the control.

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