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A BISHOP MOVEMENT ON A CHESS BOARD AS A RATIONAL FUNCTION GENERATE A LAURENT SERIES THAT IS ALGEBRAIC

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ABSTRACT

The purpose of this study was carried to examine the counting problems of non-attacking bishop placements on any chess board. The chess board for bishop placement with forbidden positions was considered as a diagonal movement to the right or to the left. We discussed the general movement of a bishop as a rational function of a power series on a chess board in variables s and t. Furthermore, we constructed the movement techniques of a bishop placements as rational function that generates a Laurent series that is algebraic. Finally, we applied it to combinatorial problems that generates' the diagonal movement of a bishop to give the Laurent series that is algebraic over k (t).

Key words: Chess movements; Laurent series; Permutation; Power Series; Handamard product.

1. INTRODUCTION

Combinatorics as a branch of mathematics has drawn much attention still to be research on due to its vast application potentials. Enumerative permutation for either the bishop or rook polynomial of any board belong to classical combinatorics and its applications (RIORDAN, 1980; Riordan, 1958) However, the contributions of this paper fall within the general area of combinatorics, a branch of mathematics involving the study of finite structures. Aspects of combinatorics include the following (Bogart, 1990; Bona, 2007; David, 2017; Stanley, 2000):

- Enumerative combinatorics: counting the structures of a given kind and size,
- Combinatorial designs and matroid theory: deciding when certain criteria can be met, and constructing and analyzing elements meeting the criteria,
- External combinatorics and combinatorial optimization: finding "largest", "smallest", or "optimal" elements,
- Algebraic combinatorics: studying combinatorial structures arising in an algebraic context, or applying algebraic techniques to combinatorial problems.

Although, the notion of a "rook polynomial" first appeared in cell decomposition technique of Riordan on a board of chess game. Notwithstanding that first case where mathematical questions were studied with the help of permutation occurred around 1770, when Joseph Louis Lagrange, in the study of polynomial equations, observed that properties of the permutations of the roots of an equation are related. Nevertheless, studies have shown that either the bishop or rook polynomial of any board can be computed recursively (Laisin, 2018; Abigail, 2004; RIORDAN, 1980) using acell decomposition technique of Riordan. The counting problems of non-attacking bishop or rook placements in the game of chess and its movement to the direction either to the right or to the left; upwards or downwards to capture pieces in the same direction with the bishop or rook respectively. However, the series of papers by (Joichi, Goldman, & White, 1978; Goldman, Joichi, & White, 1977; Goldman., Joichi, Reiner, & White, 1976; White, Goldman, & Joichi, 1975) have expanded the field by applying more advanced combinatorial methods. More recently, (LAISIN, 2018; Laisin, 2018; Michaels, 2013) and (Ono, Haglund, & Sze, 1998; Haglund, 1996) also made self-standing studies in various connections to either bishop or rook polynomials to other parts of mathematics: enumeration over finite fields, group representation theory and hypergeometric series. Furthermore, Combinatorics now consists of the following subfields and approaches: Enumerative Combinatorics, Analytic Combinatorics, Partition Theory, Graph Theory, Design Theory, Finite Geometry, Order Theory, Matroid Theory, External Combinatorics, Probabilistic Combinatorics, Algebraic Combinatorics, and Combinatorics on Words, Geometric Combinatorics, Topological Combinatorics, Arithmetic Combinatorics and Infinitary Combinatorics.

Bishop polynomial play an important role in the theory of permutations with forbidden positions.

However, if B is a board of size $n \times n$ then, the bishop polynomialof board B is denoted as $\mathfrak{B}(x, B)$ for the number of ways to place n non-attacking bishops on the board.The other pieces follow different rules for movement, but our study will focus on the strength of a bishop movements.

Furthermore, authors have shown that either the bishop or rook polynomial of any board can be computed recursively. In this paper, we present a construction for the movement techniques of a bishop placements in the game of chess that generates a Laurent series that is algebraic.

2. PRELIMINARIES

Definition 2.1.1A bishop is a chess piece that moves diagonally and capture a piece if that piece rests on a square in the same diagonal (Chung & Graham, 1995; Goldman., Joichi, Reiner, & White, 1976; White, Goldman, & Joichi, 1975; Laisin, 2018).

a. Board: A board B is an $n \times n$ array of n rows and m columns. When a board has a darkened square, it is said to have a forbidden position.
b. Bishop polynomial: A bishop polynomial on a board B, with forbidden positions is denoted as $\mathfrak{B}(x, B)$, given by

$$\mathfrak{B}(x, B) = \sum_{i=1}^{k} b_i(B) x^i,$$

where $\mathfrak{B}(y, B)$ has coefficients $b_i(B)$ representing the number of ways to place n non-attacking bishops on the board B.

Definition 2.1.2

Let B be an $n \times n$ board and its diagonal denoted by $\mathfrak{D}^{\theta}$ and suppose that;

$$F(x_1, x_2, \dots, x_k) = \sum f(n_1, n_2, \dots, n_k) x_1^{n_1} x_2^{n_2} \dots x_k^{n_k} \in K[[(x_1, x_2, \dots, x_n)]]$$

Then, the $\mathfrak{D}^{\theta}$ is the power series in a single variable x defined by

$$\mathfrak{D}^{\theta} = \mathfrak{D}^{\theta}(x) = \sum_{n} f(n, n, \dots, n) x^n$$

Definition 2.1.3

A ring R is a set with two laws of composition $+$and $\times$ called addition and multiplication, which satisfy these axions;

a. With the composition $+, R$ is an abelian group, with identity denoted by 0. This abelian group is denoted by R^{+}
b. Multiplication is associative and has an identity denoted by 1.
c. Distributive law for all $a, b, c \in R, \Rightarrow (a+b)c = ac + bc$ & $c(a+b) = ca + cb$(Artin, 1991)

A chess board B of a ring is a chess board which is closed under the operations of addition subtraction, and multiplication and which contains the first placement $(b_0(B) = 1)$.

A bishop polynomial with forbidden positions is denoted as $\mathfrak{B}(x, B)$, given by

$$\mathfrak{B}(x, B) = \sum_{i=1}^{k} b_i(B)x^i,$$

where $\mathfrak{B}(x, B)$ has coefficients $b_i(B)$ representing the number of ways of bishop's placements on B. Furthermore, on m $\times n$ board B, we have $b_0(B) = 1$ and the coefficients are determined by

$$\mathfrak{B}(x, B) = \sum_{k=0}^{\min(m,n)} \frac{n!\, m!}{k!\,(n-k)!\,(m-k)!} x^k$$

$(\text{Laisin}, 2018)$

Nevertheless, with the limitation that, "bishops must not attack each other" is removed, in this case one must choose any k squares from $m \times n$ arrays. Then, we have;

$\binom{mn}{k} = \frac{(mn)!}{k!(mn-k)!}$ ways.

Suppose that $m \neq n$, then, the k bishops will differ in some way from each other, however, the results obtained will be multiplied by $k!$, for the k bishops. Then, we have;

$\binom{m}{k}\binom{n}{k} k! = \frac{n!m!}{k!(n-k)!(m-k)!}$(Vilenkin, 1969; LAISIN, 2018).

Definition 2.1.4 (Power series)

A series having the form

$$a_0 + a_1 + a_2x^2 + a_3x^3 + \cdots + a_kx^k + \cdots \sum_{k=0}^{\infty} a_kx^k$$

Where a_k are constant coefficients of the power series with a single variable *x*.

Lemma 2.2

If B is a board of darkened squares that decomposes into two disjoint sub-boards$B_i : i = 1 \; and \; 2$, then$\mathfrak{B}(x, B) = \mathfrak{B}(x, B_1)\mathfrak{B}(x, B_2)$(White, Goldman, & Joichi, 1975; Laisin, 2018).

Lemma 2.3

Suppose $S \in K_{rat}((X))$ and $T \in K_{alg}((X))$ then $S * T \in K_{alg}((X))$(Banchoff, 1991)

Theorem2.4

Suppose, B isan $n \times n$board ofdarkenedsquares with bishops that move through a direction of an angle of θ^0then, $\mathfrak{B}(x, B)$ for the n-disjointsub-boards is;

$$\mathfrak{B}(x, B) = \sum_{i=0}^{n} \prod_{k=0}^{n} \mathcal{X}_{B_{j,K}}(x)^i b_i^{\theta}(B_j), \;\; j = 1,2, \dots n. \quad (\text{Laisin}, 2018)$$

Theorem 2.5 (The Fundamental Right Diagonal Theorem)
Let B be a chessboard containing a right diagonal ($\theta = 45^0$)with k cells. Let $B - \theta_{(i,j)}$be the board obtained from B byremoving right diagonal i and left diagonal j (i.e. one of the k left diagonals containing acell in the right diagonal i). Then

$$\mathfrak{B}(x, B) = \mathfrak{B}(x, B - \theta_i) + x \sum_{i=0}^{n} \mathfrak{B}\left(x, B - \theta_{i,j}\right)$$

(Shanaz , 1999)

Theorem 2.6
Let$T \epsilon K_{alg}((X))$ where X is a finite alphabet. Then $\emptyset(S)$ is algebraic over the field $K(X)$ of rational functions in X (Banchoff, 1991).

3. Main Results

Theorem 3.1
Let $\mathfrak{D}^{\theta}$be a power series on a chess $n \times n$ board in variables s and t that represent the movement of a bishop as a rational function in $K(t)$. Then $\mathfrak{D}^{\theta}$ is algebraic. Hence, the Laurent series $(\alpha)S(X, X^{-1})$ is algebraic over $K(t)$.

Proof
Let$\mathfrak{B}(s, t) \in K((s, t)) \cap K(s, t)$ and assume that K is algebraically closed. Then, the Laurent series $\omega \in K((t))$ is also algebraic.
Now, consider the series

$$S(x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k) \in K_{rat}(t)((X, Y))$$

Thus, the rational series S is in X and Y with coefficient in the field $K(t)$ of rational function in one variable t.

Implies $K_{rat}(t) \subset K_{alg}(t)$

However, the coefficients of s is a Laurent series in t (where t is a bishop movement in $\mathfrak{D}^{\theta}$)
Now assume that the series

$$S(X, X^{-1}) = S(x_1, x_2, \ldots, x_k, x_1^{-1} x_2^{-1}, \ldots, x_k^{-1})$$

$S(X, X^{-1})$ is well defined and the coefficients of a Laurent monomial $\alpha = (x_{i_1}^{a_1} x_{i_2}^{a_2}, \ldots, x_{i_j}^{a_j})$ for all $(a_1, a_2, \ldots, a_j) \in \mathbb{Z}$ is a formal Laurent series $\omega \in K\,(t)$ in the variable t.
Since

$$(\alpha)S(X, X^{-1}) = (1)\; T\; (X, X^{-1})$$

where $T = \alpha^{-1}\, S\,(X, Y\ )$ and T is rational when $S(X, Y)$ is always rational without any loss of generosity we can assume that $\alpha = 1$
Since,

$$X = (x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k)$$

and let $\sigma \subset X^*$ be the set of all movements of X^* that reduce to the identity $b_0(B) = 1$ under the relations

$$x_i y_i = y_i x_i = 1 \qquad for\ all\ \ 1 \le i \le k$$

Nevertheless, the coefficient we need is the sum of all coefficients of the Hadamard product $S * \sigma$ and the abelianization $\emptyset(S * \sigma)$.
Since
$S \in K_{rat}(t)((X, Y))$ and $\sigma \in K_{alg}(t)\big((X, Y)\big) \subset K_{alg}(t)((X, Y))$

Then, by lemma 2.3 we have

$$S * \sigma \in K_{alg}(t)((X, Y))$$

Thus, by theorem 2.6 $\emptyset(S * \sigma)$ is algebraic over $K(t)((X, Y))$. By lemma 2.3, we have $\emptyset(S * \sigma)_{x_i=y_i=1}$ is algebraic over $K(t)$. Hence, the Laurent series $(\alpha)S(X, X^{-1})$ is algebraic over $K(t)$.

4. NUMERICAL APPLICATIONS

Example

Suppose the diagonal movement is a power series

$$\mathfrak{D}^{\theta} = x_1 + x_2 + \cdots + x_k + x_1^{-1} + x_2^{-1} + \ldots + x_k^{-1}$$

That is a non-commutative Laurent polynomial over k with variable $X = (x_1, x_2, \ldots, x_k)$. Let $\alpha \in F_k$ be noncommutative Laurent monomial X. Show that the power series

$$y = \sum_{n=0}^{\infty} [\alpha]\left(\mathfrak{D}^{\theta^n}\right) t^n \text{is algebraic}$$

Solution

$\mathfrak{D}^{\theta} \in K(F_k)$for all $k = 1, 2, \cdots, k$ the group algebra of the free group F_k generated by X. Now consider $y = [\alpha]\left(1 - \mathfrak{D}^{\theta^n} t\right)^{-1}$, then since;

$\mathfrak{D}^{\theta} = x_1 + x_2 + \cdots + x_k + x_1^{-1} + x_2^{-1} + \ldots + x_k^{-1}$ and $\alpha = 1$ then

$$y = \sum_{n=0}^{\infty} [1](x_1 + x_2 + \cdots + x_k + x_1^{-1} + x_2^{-1} + \ldots + x_k^{-1})^n t^n$$
$$= 1 + 2kt^2 + (8k^2 - 2k)t^4 + (40k^3 - 24k^2 + 4k)t^6 + \cdots$$

Thus, by applying combinatorial argument, we have;

$$y = \frac{(2k-1)k\sqrt{1 - 4(2k-1)t^2}}{1 + (k-1)k\sqrt{1 - 4(2k-1)t^2}}$$

CONCLUSION

Bishop polynomials are not just interesting for their own sake. They have a variety of applications because they directly relate to permutations with restricted positions. In addition, we presented a construction for the movement techniques of a bishop placements in the game of chess that generates the Laurent series $(\alpha)S(X, X^{-1})$ that is algebraic over $K(t)$.

RECOMMENDATIONS

Further study can be carried out with the bishops on three-dimensional boards. In addition, further studies could also examine what would happen to rook and bishop polynomials by allowing the rook or bishop to attack each other on the chess boards.

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